Specific Ionic Velocities.

I. Aqueous Solutions.

Ion.	Velocity observed.	Velocity calculated from Kohlrausch's theory.
Copper	0·00026* 0·000309	0·00031
Chlorine	0·00057* 0·00059*	0·00053
Bichromate group (Cr_2O_7)	0·00048 0·00047 0·00046	0·000473

II. Alcoholic Solutions.

Salt.	Velocity of anion (observed).	Velocity of kation (observed).	Sum of velocities (observed).	Sum of velocities (calculated).
Cobalt chloride	0·000026	0·000022	0·000048	0·000060
Cobalt nitrate	0·000035	0·000044	0·000079	0·000079

II. "Memoir on the Theory of the Compositions of Numbers." By P. A. MacMahon, Major R.A., F.R.S. Received November 17, 1892.

(Abstract.)

In the theory of the partitions of numbers the order of occurrence of the parts is immaterial. Compositions of numbers are merely partitions in which the order of the parts is essential. In the nomenclature I have followed H. J. S. Smith and J. W. L. Glaisher. What are called "unipartite" numbers are such as may be taken to enumerate undistinguished objects. "Multipartite" numbers enumerate objects which are distinguished from one another to any given extent; and the objects are appropriately enumerated by an ordered assemblage of integers, each integer being a unipartite number which specifies the number of objects of a particular kind; and such assemblage constitutes a multipartite number. The 1st

* Preliminary determinations.

Section treats of the compositions of unipartite numbers both analytically and graphically. The subject is of great simplicity, and is only given as a suitable introduction to the more difficult theory, connected with multipartite numbers, which is developed in the succeeding sections.

The investigation arose in an interesting manner. In the theory of the partitions of integers, certain partitions came under view which may be defined as possessing the property of involving a partition of every lower integer in a unique manner. These have been termed "perfect partitions," and it was curious that their enumeration proved to be identical with that of certain expressions which were obviously "compositions" of multipartite numbers.

The 2nd Section gives a purely analytical theory of multipartite numbers.

$$\overline{p_1 p_2 p_3 \dots p_n}$$

is the notation employed in the case of the general multipartite number of order n. The parts of the partitions and compositions of such a number are themselves multipartite numbers of the same order. Of the number $\overline{21}$ there exist

Partitions.	Compositions.
$(\overline{21})$	$(\overline{21})$
$(\overline{20}\ \overline{01})$	$(\overline{20}\ \overline{01})$, $(\overline{01}\ \overline{20})$
$(\overline{11}\ \overline{01})$	$(\overline{11}\ \overline{10})$, $(\overline{10}\ \overline{11})$
$(\overline{10}^2\ \overline{01})$	$(\overline{10}\ \overline{01})$, $(\overline{10}\ \overline{01}\ \overline{10})$, $(\overline{01}\ \overline{10}^2)$.

The generating function which enumerates the composition has the equivalent forms

$$\frac{h_1+h_2+h_3+\dots}{1-h_1-h_2-h_3-\dots},$$

$$\frac{a_1-a_2+a_3-\dots}{1-2(a_1-a_2-a_3-\dots)},$$

where h_s, a_s represent respectively the sum of the homogeneous products of order s and the sum of the products s together of quantities

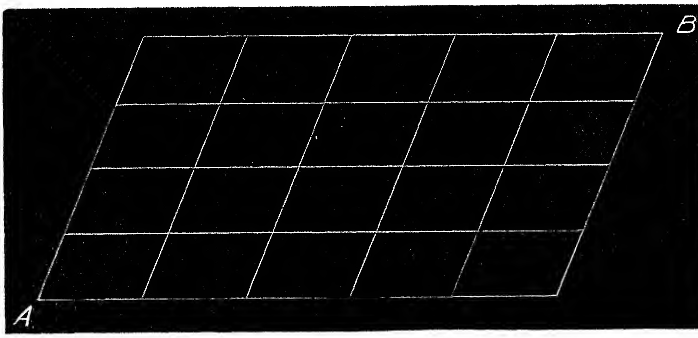
$$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n,$$

and the number of compositions of the multipartite

$$\overline{p_1 p_2 \dots p_n}$$

is the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ in the development according to ascending powers.

Section 3 is taken up with the graphical representation of bipartite numbers. A reticulation is formed which consists of a series of points through each of which straight lines pass in two definite directions, the boundary of the whole being a parallelogram.



The figure AB is the graph of the number $\overline{54}$. A composition of this number is defined by fixing nodes at certain points which possess the property that no point is at once above and to the left of any other point; the parallelogram between adjacent nodes is the graph of a certain number, and in passing through the nodes in succession from A to B an ordered assemblage of numbers is found which constitutes a composition of the number which is represented by the whole graph.

This conception leads to theorems of a new kind which are generalised in Section 4 to include tripartite and multipartite numbers. This section is the most important part of the investigation. It is established that

$$\frac{1}{2}\,\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

is also a generating function which enumerates the compositions; the coefficient of

$$s_1^{p_1}s_2^{p_2}\ldots s_n^{p_n}\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

being the number of compositions possessed by the multipartite

$$\overline{p_1p_n\ldots p_n}.$$

The generating function of the previous section 2 may, by the addition of the fraction $\frac{1}{2}$ and the substitution of $s_1\alpha_1$, $s_2\alpha_2$, &c., for α_1, α_2, &c., be thrown into the form

$$\frac{1}{2}\,\frac{1}{1-2\left(\Sigma s_1\alpha_1-\Sigma s_1s_2\alpha_1\alpha_2+\ldots(-)^{n+1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n\right)},$$

and hence these two fractions, in regard to the terms in their expansions which are products of powers of $s_1\alpha_1, s_2\alpha_2, \ldots, s_n\alpha_n$, must be identical. This fact is proved by means of the identity—

$$\tfrac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

$$=\tfrac{1}{2}\frac{1}{1-2(\Sigma s_1\alpha_1-\Sigma s_1s_2\alpha_1\alpha_2+\ldots(-)^{n+1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n)}$$

multiplied by

$$1+\Sigma\frac{2(A_{\kappa_1}+\alpha_{\kappa_1})\ldots(A_{\kappa_t}+\alpha_{\kappa_t})-(A_{\kappa_1}+2\alpha_{\kappa_1})\ldots(A_{\kappa_t}+2\alpha_{\kappa_t})}{(1-S_{\kappa_1})\ldots\ldots\ldots\ldots(1-S_{\kappa_t})}s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_n},$$

where

$$S_\kappa = s_\kappa(2\alpha_1+\ldots+2\alpha_\kappa+\alpha_{\kappa+1}+\ldots+\alpha_n) = s_\kappa(A_\kappa+2\alpha_\kappa),$$

and the summation is in regard to every selection of t integers from the series

$$1, 2, 3, \ldots n,$$

and t takes all values from 1 to $n-1$.

This remarkable theorem leads to a crowd of results which are interesting in the theory of numbers. One result in the pure theory of permutations may be stated.

Calling a contact $\alpha_u\alpha_t$ a *major* contact when $u > t$, the number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

which possess exactly s major contacts is given by the coefficient of

$$\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the product

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n},$$

and, moreover, is equal to the number of permutations for which

$$r_2+r_3+\ldots+r_n = s,$$

r_t denoting the number of times that the letter α_t occurs in the first

$$p_1+p_3+\ldots+p_{t-1}$$

places of the permutation.

Section 5 gives an extension of the idea of composition and of the foregoing theorems.

The geometrical method of "trees" finds here a place, and, lastly, there is the fundamental algebraic identity—

$$\frac{1}{k}\frac{1}{\left\{1-s_1\left(ka_1+a_2+\ldots+a_n\right)\right\}\left\{1-s_2\left(ka_1+ka_2+\ldots a_n\right)\right\}\ldots\left\{1-s_n\left(ka_1+ka_2+\ldots+ka_n\right)\right\}}$$

$$=\frac{1}{k}\frac{1}{1-k\Sigma s_1a_1+k\left(k-1\right)\Sigma s_1s_2a_1a_2-\ldots+\left(-\right)^nk\left(k-1\right)^{n-1}s_1s_2\ldots s_na_1a_2\ldots a_n}$$

multiplied by

$$1+\Sigma\frac{k\left(A_{t_1}+a_{t_1}\right)\ldots\left(A_{t_n}+a_{t_n}\right)-\left(A_{t_1}+ka_{t_1}\right)\ldots\left(A_{t_n}+ka_{t_n}\right)}{\left(k-1\right)\left(1-S_{t_1}\right)\left(1-S_{t_2}\right)\ldots\left(1-S_{t_n}\right)}s_{t_1}s_{t_2}\ldots s_{t_n},$$

which reduces to that formerly obtained when k is given the special value 2.

Presents, November 24, 1892.

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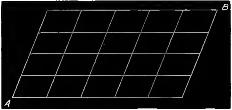
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